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Suppression and enhancement of diffusion in disordered dynamical systems

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The impact of quenched disorder on deterministic diffusion in chaotic dynamical systems is studied. As a simple example, we consider piecewise linear maps on the line. In computer simulations we find a complex scenario of multiple suppression and enhancement of normal diffusion, under variation of the perturbation strength. These results are explained by a theoretical approximation, showing that the oscillations emerge as a direct consequence of the unperturbed diffusion coefficient, which is known to be a fractal function of a control parameter.

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Recently there has been growing interest in the field of *disordered dynamical systems*, thus trying to bring together two, at first view, very different directions of research [1,2]: Diffusion on disordered lattices with quenched (static) randomness is a traditional problem of statistical physics, which can be studied by probabilistic methods being developed in the theory of stochastic processes [3–5]. However, diffusion can also be generated from deterministic chaos in nonlinear equations of motion [6,7] making it possible to assess chaotic and fractal properties of diffusion by methods of dynamical systems theory [8–10]. Hence, understanding the dynamics of disordered dynamical systems poses the challenge of suitably combining these different concepts and ideas. To our knowledge, only very few cases of respective models have been studied so far. Examples include random Lorentz gases for which Lyapunov exponents have been calculated by means of kinetic theory and by computer simulations [10,11], numerical studies of diffusion on disordered rough surfaces and in disordered deterministic ratchets [12], as well as numerical and analytical studies of chaotic maps on the line with quenched disorder [2,13].

In this work we will focus on the most simple example in the latter class of models, which are piecewise linear maps on the line. In case of mixing dynamics, unperturbed maps of this type exhibit normal diffusion [7,14–17]. However, adding quenched disorder in the form of a local bias with globally vanishing drift profoundly changes the dynamics, leading to subdiffusion in a complicated potential landscape [2,13]. Here we will consider a different type of static randomness, which is multiplicative, preserves the local symmetry of the model, and is not related to Levy distributions [18], thus not resulting in anomalous diffusion. Consequently, here we denote with suppression and enhancement of diffusion, the variation of the *normal* diffusion coefficient. Another important aspect is that in previous work the disordered maps were always exhibiting the Bernoulli property [2,13], therefore the diffusive properties were in agreement with expectations from stochastic theory. In our case we start from an unperturbed model that is known to exhibit strong dynamical correlations, resulting in a fractal diffusion coefficient as a function of the control parameter [15–17]. Adding uncorrelated static randomness enables to study in which way these dynamical correlations survive, or are getting destroyed, as a function of the perturbation strength, and to which extent simple random walk theory may still be applicable for understanding perturbed chaotic diffusion.

The unperturbed model is defined by the equation of motion

$$x_{n+1}=M_a(x_n), \tag{1}$$

where $a\in R$ is a control parameter and x_n is the position of a point particle at discrete time n. $M_a(x)$ is continued periodically beyond the interval $[-1/2,1/2)$ onto the real line by a lift of degree 1, $M_a(x+1)=M_a(x)+1$. We assume that $M_a(x)$ is antisymmetric with respect to $x=0$, $M_a(x)=-M_a(-x)$. The map we study as an example is defined by $M_a(x)=ax$, where the uniform slope a serves as a control parameter. The Lyapunov exponent of this map is given by $\lambda(a)=\ln a$ implying that for $a>1$ the dynamics is chaotic. We now modify this system by adding a random variable $\Delta a(i)$, $i\in Z$, to the slope on each interval $[i-1/2,i+1/2)$ yielding $M_a^{(i)}(x)=[a+\Delta a(i)]x$. We assume that the random variables $\Delta a(i)$ are independent and identically distributed according to a distribution $\chi_\alpha(\Delta a)$, where α is again a control parameter. In the following we will consider two different types of such distributions, namely, random variables distributed uniformly over an interval of size $[-\alpha,\alpha]$,

$$\chi_\alpha(\Delta a)=\Theta(\alpha+\Delta a)\Theta(\alpha-\Delta a)/(2\alpha), \tag{2}$$

and dichotomous or δ-distributed random variables,

$$\chi_\alpha(\Delta a)=[\delta(\alpha+\Delta a)+\delta(\alpha-\Delta a)]/2. \tag{3}$$

Since $|\Delta a|\leq\alpha$, we denote α as the perturbation strength. As an example, we sketch in Fig. 1 the map resulting from the disorder of Eq. (2) as applied to the slope $a=3$. In the absence of any bias, the diffusion coefficient is defined as $D(a,\alpha)=\lim_{n\to\infty}\langle x_n^2\rangle/(2n)$, where the brackets denote an ensemble average over moving particles for a given configuration of disorder. An additional disorder average is not necessary because of self-averaging [4]. Note that for locally symmetric quenched disorder and $(a-\alpha)>2$, there is no

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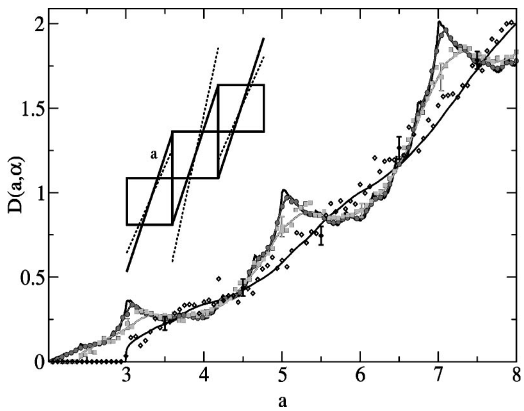

FIG. 1. Diffusion coefficient $D(a,\alpha)$ for the piecewise linear map shown in the figure. The slope a is perturbed by static disorder of maximum strength α as defined in Eq. (2). The bold black line depicts $D(a,0)$. The symbols represent computer-simulation results for $\alpha>0$, the corresponding lines are obtained from the theoretical approximation, Eq. (5). The parameter values are $\alpha=0.1$ (circles), $\alpha=0.4$ (squares), and $\alpha=1.0$ (diamonds).

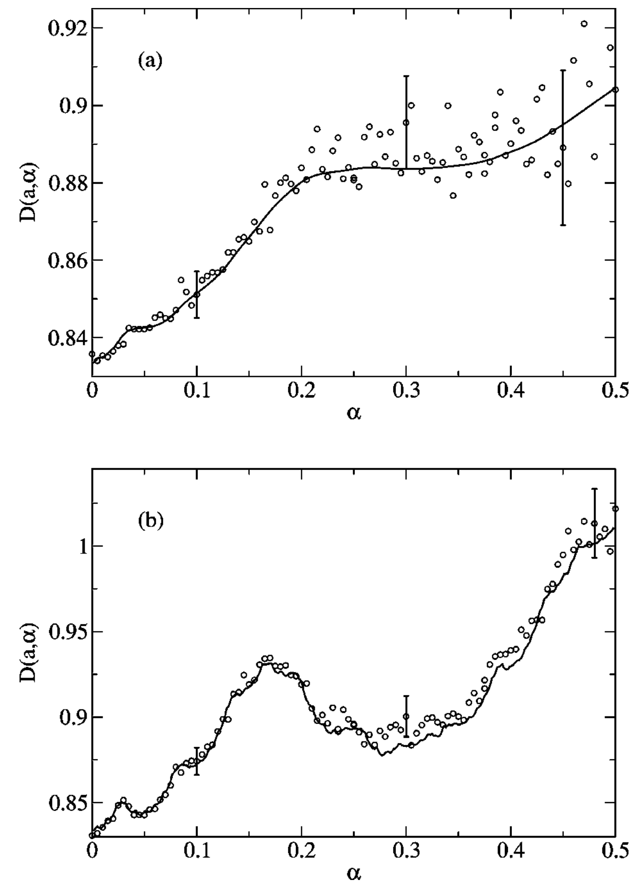

FIG. 2. Diffusion coefficient $D(a,\alpha)$ as a function of the perturbation strength α at slope $a=6$: (a) disorder distribution, Eq. (2), (b) disorder distribution, Eq. (3). The circles represent results from computer simulations, the lines are obtained from the approximation, Eq. (5).

physical mechanism leading to infinitely high reflecting barriers as they are responsible for Golosov localization [2,4]. Thus diffusion must be normal, as is confirmed by computer simulations. Hence, the central question is what happens to the parameter-dependent diffusion coefficient $D(a,\alpha)$ under variation of the two control parameters a and α.

For the unperturbed case, $\alpha=0$, the diffusion coefficient has been computed in Refs. [15,16] showing that $D(a,0)$ is a fractal function of the slope a as a control parameter. This function is depicted in Fig. 1, as well as results from computer simulations for different values of the perturbation strength α in case of uniform disorder [19]. As expected, this irregular structure gradually disappears by increasing α. However, it is remarkable that even for large perturbation strength α, oscillations are still visible as a function of a indicating that the underlying dynamical correlations are very robust against this type of perturbation. Note, furthermore, the nonanalytical behavior of $D(a,\alpha)$ for $\alpha>0$ and small a, thus indicating the existence of a dynamical phase transition, which was not present in the unperturbed case.

Before we proceed to more detailed simulation results we briefly repeat what is known for diffusion in lattice models with random barriers [3–5,20]. In the most simple version, the quenched disorder is defined on a one-dimensional periodic lattice with transition rates between neighboring sites i and $i+1$ having the symmetry $\Gamma_{i,i+1}=\Gamma_{i+1,i}\equiv\Gamma_k$ for a given random distribution of Γ_k. In this situation an exact expression for the stochastic diffusion coefficient has been derived reading [5,20]

$$d=\{1/\Gamma\}^{-1}l^2, \tag{4}$$

with the brackets defining the disorder average $\{1/\Gamma\}=1/N\Sigma_{k=0}^{N}1/\Gamma_k$ at chain length N and for a distance l between sites. The double-inverse demonstrates that the highest barriers dominate diffusion in one dimension, thus $\Gamma_k\to 0$ naturally leads to a vanishing diffusion coefficient. This scenario is translated to the map under consideration as follows: Eq. (1) can be understood as a time-discrete Langevin equation, $x_{n+1}=x_n-\partial V(x_n)/\partial x_n$ [2,7]. Under proper integration of $M_a^{(i)}(x)$ the corresponding potential $V(x)$ is reminiscent of a random barrier model with the perturbation strength α determining the highest barriers. Hence, simple random-walk theory predicts suppression of diffusion for the chaotic map, viz., $D(a,\alpha)$ being a monotonical decreasing function of α.

To check this hypothesis, we choose fixed values of a corresponding to the two extreme situations of starting from a local maximum or minimum, respectively, of the unperturbed $D(a,0)$ in Fig. 1. We first focus on the local minimum at $a=6$ for $\alpha\leqslant 0.5$ with uniform and dichotomous disorder, see Fig. 2. In sharp contrast to the prediction of the simple random-walk theory outlined above, in both cases we observe enhancement of diffusion as a function of α. Moreover, this enhancement does not appear in the form of a simple functional dependence on α: In (a), smoothed-out oscillations are visible on smaller scales, whereas in (b) the resulting function is clearly nonmonotonic and wildly fluctuating, exhibiting multiple suppression and enhancement in different parameter regions. Results on larger scales of α are depicted

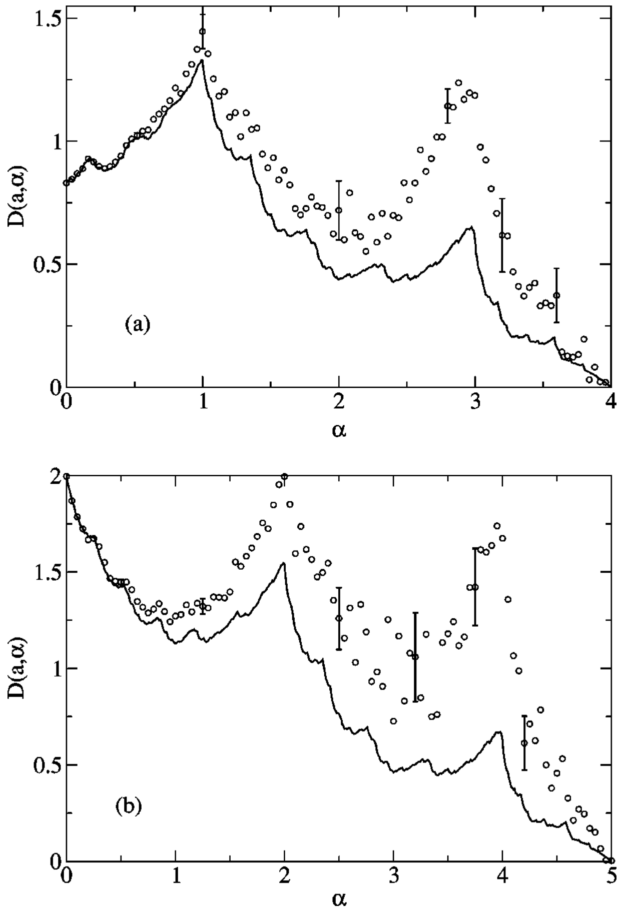

FIG. 3. Diffusion coefficient $D(a,\alpha)$ as a function of the perturbation strength α for dichotomous disorder, Eq. (3), at two different slopes a: (a) $a=6$, (b) $a=7$. The circles represent results from computer simulations, the lines are obtained from the approximation, Eq. (5).

in Fig. 3 for dichotomous disorder. Figure 3(b) shows that choosing a at a local maximum of $D(a,0)$ leads to suppression of diffusion for $\alpha<1.0$, whereas a local minimum generates enhancement in the same parameter region of α. We emphasize that in both cases the diffusion coefficient decreases on a very coarse scale by increasing α, thus recovering qualitative agreement with the simple random-walk prediction, Eq. (4) [21]. Indeed, for $(a-\alpha)\to 2$, barriers are formed that a particle cannot cross anymore implying the existence of localization.

To theoretically explain the simulation results, we modify Eq. (4) in a straightforward way such that it can be applied to our disordered deterministic map under consideration. We first note that for uniform transition rates, $\Gamma_k=$const., it is $d(\Gamma_k,l)=\Gamma_k l^2$. Using this familiar expression for the random-walk diffusion coefficient on the unperturbed lattice, we rewrite Eq. (4) as $d=\{1/d(\Gamma_k,l)\}^{-1}$. In the case of our map, the transition rates and the distance between sites are both somewhat combined in the action of the slope a as a control parameter. Therefore, the unperturbed diffusion coefficient is correctly rewritten by replacing $d(\Gamma_k,l)\equiv d(a)$. The key question is: What function shall be used for the parameter-dependent diffusion coefficient $d(a)$ in case of deterministic dynamics? Here we propose *to identify the function $d(a)$ with the exact, unperturbed deterministic diffusion coefficient* previously defined as $D(a,0)$. Providing this information, the exact formula, Eq. (4), for stochastic dynamics becomes an *approximation* that can straightforwardly be applied to deterministic dynamics in disordered systems. If the disorder distributions $\chi_\alpha(\Delta a)$ are bounded by the perturbation strength α, and by taking the continuum limit for the random variable, our final result reads

$$D_{app}(a,\alpha)=\left[\int_{-\alpha}^{\alpha} d(\Delta a)\,\frac{\chi_\alpha(\Delta a)}{D(a+\Delta a,0)}\right]^{-1}. \qquad (5)$$

This expression represents the central formula of our paper. The results obtained from it are depicted in Figs. 1 to 3 in the form of lines. For small enough α the agreement between theory and simulations is excellent, thus confirming the validity of this equation. For larger α our theory still correctly predicts the oscillations generated from dichotomous disorder; however since Eq. (5) is approximate it should not be surprising to detect quantitative deviations.

We now show that this formula provides a physical explanation for the complex dependence of the diffusion coefficient on the perturbation strength. For $\alpha\to 0$, Taylor expansion leads to

$$D_{app}(a,\alpha)=\int_{-\alpha}^{\alpha} d(\Delta a)\chi_\alpha(\Delta a)D(a+\Delta a,0). \qquad (6)$$

We remark that Eq. (6) can be proven without advocating Eq. (5), by starting from the definition of the diffusion coefficient [22]. In this limit the perturbed diffusion coefficient reduces to an average of the exact diffusion coefficient over the neighborhood $[a-\alpha,a+\alpha]$ weighted by the respective disorder distribution $\chi_\alpha(\Delta a)$. Consequently, if a is chosen at a local minimum, the result must be the enhancement of diffusion by increasing α and suppression at a local maximum, respectively [23]. On these grounds it is clear that the fractal parameter dependence of $D(a,0)$ must reappear in the perturbed diffusion coefficient, hence leading to multiple suppression and enhancement on all scales.

We conclude with a few remarks.

(a) It would be important to have a proof of Eq. (5) for dynamical systems, as well as to obtain higher-order corrections for explaining the deviations between simulation and theory as visible in Fig. 3.

(b) Our results might be important to understand diffusion on a stepped surface with a disordered arrangement of Ehrlich-Schwoebel barriers, as has been analyzed on the basis of a random trap/random barrier model [24]. Our map provides a generalization of such a model in terms of correlated random walks and thus enables to study the impact of memory effects on surface diffusion.

(c) One could think of applying our approach to systems such as those studied in Ref. [12], or to the periodic Lorentz gas [8–10], which is a model close to experiments on antidot lattices [25]. Knowing the density-dependent diffusion coefficient in the unperturbed case [26] leads us to predicting local and global suppression of the diffusion coefficient in this model in case of static density fluctuations.

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